Review of GIANDOMENICO SICA (ed.) ***What is Category Theory?*** Polimetrica, 2006

For *Studia Logica*

John Symons

University of Texas at El Paso

Giandomenico Sica's volume is a collection of eleven papers on category theory by philosophers, mathematicians, and mathematical physicists. In addition to papers of direct interest to philosophers of mathematics, the volume contains some introductory expositions of category theory along with a valuable discussion of the relationship between category theory and physics by Bob Coecke. While there are several technically difficult papers, the volume as a whole is reasonably accessible to those with some familiarity with the basics of category theory. The importance of the volume lies in the possibility that it will encourage broader interest in category theory among philosophers.

Category theory is a branch of abstract algebra devoted to investigating transformations in a highly abstract form. In his excellent recent textbook, Steve Awodey characterizes category theory as the "mathematical study of (abstract) *algebras of functions.* Just as group theory is the abstraction of the idea of a system of permutations of a set of symmetries of a geometric object, category theory arises from the idea of a system of functions among some objects." (2006, 1) While there is obviously a long history of reflection on the idea of transformation in geometry and algebra, the development of abstract algebra in the 1930s permitted the study of transformations and compositions of transformations in the most general form possible.

In the first essay in this volume, ‘Abstract and Variable Sets in Category Theory’, John Bell argues that the categorical treatment of sets unifies continuity and discreteness. This is a deep and interesting article and a great entry point into category theory for philosophers. Bell’s paper is useful as an introduction to the category theoretical way of thinking about mathematical problems. He explains that category theory does not define transformation in terms of some other notion as might be the case, for instance, in a set theoretic approach. This paper serves as a good starting point for the volume, insofar as it compares set theoretic accounts of property, attribute and definition with those that that figure in the categorial approach to sets. Some of the main themes of category theory are introduced via Bell’s discussion of abstract sets. Abstract sets are introduced by contrast with concrete sets. Cantor described sets as collections of definite, “well-differentiated objects of our intuition or thought.” (1895) Bell calls sets of this kind *concrete sets* by contrast with the notion of an *abstract set* which he also draws from Cantor’s work (but this time from Cantor’s account of cardinality). An abstract set is what we are left with when each element “is purged of all intrinsic qualities *aside from the quality which distinguishes that element from the rest.”* (10) Such a set is distinguishable from others solely by virtue of the number of elements of the set, leading Bell to call an abstract set “an image of pure discreteness” and “an embodiment of raw plurality”.

Ordinarily in set theory, membership in the set is usually fixed by reference to something like the possession of an attribute or the instantiation of some property; membership in the concrete set is fully determined by the relevant attribute. Concrete set theory can be understood to be the theory of the extension of attributes. An *abstract* set, by contrast, is not the extension of an attribute. The only characteristic that the members of abstract sets possess is distinguishability from other elements. Given the failure of the object/attribute distinction we can see that the membership relation also does not hold as a primitive within abstract set theory. The only remaining property of interest in an abstract set is the discreteness of the objects in that set. The principles governing what Bell calls abstract set theory are derived from this "fact". Relations between abstract sets are entirely reducible to relations between their constituting elements. Bell's description of the process of abstraction from sets, members of sets, and properties of sets to the relations that characterize sets in their most general form gives readers who are familiar with set theory a feel for the rarefied world of category theory.

Category theory can be understood as an abstract algebra of relations, mappings, or functions. Having something like this is interesting to philosophers and mathematicians for a number of reasons. It provides a toolbox of techniques whereby relationships between distinct domains of mathematical investigation can be illuminated. The first presentations of category theory arose out of algebraic topology and specifically with Samuel Eilenberg's observation that Sanders MacLane's calculations on a specific case of a group extension coincided precisely with Norman Steenrod's calculation of the homology of a solenoid. This history is discussed in detail in the piece by Jean Pierre Marquis in the Sica volume. Eilenberg and MacLane's effort to make sense of this coincidence across apparently distinct areas of mathematical inquiry gave rise to their development of category theory. It should be of obvious interest to philosophers that there are direct connections between fields like set theory and proof or logic and geometry which have been described in categorial terms. One striking example, as mentioned by Awodey in his textbook, involves the categorial notion of an adjoint functor which occurs in logic as the existential quantifier and in topology as the image operation along a continuous function. (2006, 2) Category theory offers all kinds of deep and surprising insight into the shared features of a wide variety of phenomena. So, clearly this is an intrinsically interesting field of study for philosophers.

There are an increasing number of introductory texts which can provide philosophers some technical acquaintance with category theory (See for example Lawvere and Schanuel 1997; Awodey 2006; Peirce 1991). Furthermore, Awodey and Reck provide an excellent historical account of the place of category theory in the development of formal philosophy in the twentieth century in their 2002 articles. Given the increasing accessibility of category theory to philosophers it is likely that there will be more interesting engagement with the field in the years ahead.

Given its generality and applicability, category theory is widely thought to serve as a viable alternative to set theory as a foundation for mathematics. Not surprisingly, most of the work on category theory that has been done by philosophers has focused relatively narrowly on questions pertaining to the role of category theory in the foundations of mathematics and its relationship to set theory. This focus comes largely in response to the claims of category theorists to have identified the elementary topoi that are equivalent to the category **Set**, thereby effectively axiomatizing set theory in categorial terms.[1] (see Mitchell 1972 and Cole 1973) The foundational status of category theory is one of the central concerns of contributors to the Sica volume, but it is certainly not the only topic discussed.

---

[1] A good way to get some fluency in the categorial approach to logic and set theory is via Robert Goldblatt's excellent book which has recently been reprinted by Dover. (2006)

I will not attempt to summarize or evaluate all the papers in this volume individually. However, from a philosopher's perspective the ninth paper in the volume, by Jean Pierre Marquis', is the most ambitious in the collection. This paper could serve as a good starting point for philosophers who are new to category theory and who are interested in understanding why this area of investigation is of interest. Marquis provides a sophisticated historical discussion of the development of category theory while claiming that category theory is superior to set theory for work in the foundations of mathematics. Marquis is impressed by the power of category theory to reveal the systematic similarities across different areas of mathematical inquiry and discusses some of its prominent successes. He demonstrates category theory's capacity to show how broad similarities across domains are related to one according to simple and general principles. Finally, he suggests that category theory provides "an overall conceptual frame" for mathematics and concludes by saying that category theory is "the architectonic of concepts or of conceptual systems in general." (252) This paper, along with Marquis' excellent entry in the *Stanford Encyclopedia of Philosophy*, should be enough to motivate curious philosophers to delve deeper.[2]

Marquis' enthusiasm is not shared by all the contributors to this volume. Solomon Feferman, provides an excellent summary of his previous work on the foundational status of category theory in which he argues that unlike set theory, category theory has failed to deal successfully with the problems surrounding objects that are "too large". Objects like the category of all categories, Feferman argues, pose problems which require precisely the strategies that one finds the general framework of axiomatic set

[2] Marquis' entry in the *Stanford Encyclopedia of Philosophy* can be found here: http://plato.stanford.edu/entries/category-theory/ (last retrieved October 22, 2007)

theory. He cites Mac Lane (1961, 1971) and Grothendieck (1962) as instances of this approach and notes that Vidyānāth Rao's (2006) work provides evidence to the same effect. Rao's paper in this volume 'On Doing Category Theory within Set Theoretic Foundations' describes some of the ways that set theory can be modified in order to make it adequate to the purposes of category theory. While Feferman has his own objections to set theory, (1998) he has long recommended set theoretic solutions to foundational problems in category theory. In Feferman and Rao's papers, readers will find compelling reasons to look before we leap into categorial foundations of mathematics.

It is important for a volume like this one to air some of the concerns regarding the foundational status of category theory. If, as Rao and Fefermann point out, the foundations of category theory are in need of set theoretic reinforcement, then this should make us rethink precisely what it is that we think category theory is good for.

Marquis' paper makes the case for the power of category theory at the conceptual or epistemological level. However, one of the most striking features of category theory is its potential as a formal means for exploring metaphysical or ontological questions. Beyond the epistemological realm, category theory may hold promise as a means of characterizing the most general features of individuation and transformation in formal terms. (See Symons et.al 2007)

Whatever foundational role it might ultimately play in the philosophy of mathematics, as Coecke's paper makes clear, category theory can already be elegantly integrated into the practice of quantum physics and quantum informatics. Additionally, the essential role that category theory has played in quantum topology is represented in this volume by J. Scott Carter's paper. He provides an overview of the categorial structures that are associated with embedded surfaces in 3-space and knotted surfaces in 4-space. Carter's is probably the most challenging paper in the volume for the non-physicist/mathematician.

A volume of this kind requires a good introduction in order to orient the reader with respect to the wide variety of kinds of articles and levels of technicality. Sadly, this book has none. While virtually all the contributors to the volume are well-known, it would have been useful to know something about Sica's reasons for inviting this particular group of scholars. Notably absent, for example is Steve Awodey.

The publisher, *Polimetrica*, is responsible for some of the book's most obvious flaws. To begin with, this volume should have been produced with an index. This would have permitted readers to straightforwardly contrast the views of the contributors with respect to some of the key notions of category theory. Additionally, the book would have benefitted from even a cursory copy-editing and proof-reading process. The English is sometimes shaky, there are terrible stylistic problems with some of the papers and there are numerous typos in the book. Finally, this is a book which deserves a wide audience. It is therefore regrettable and very puzzling to find that the book is not available via any of the major online book sellers at this time.

Nevertheless, great credit is due to Sica for organizing this interesting and potentially very important volume. In addition to covering the debate concerning the foundational status of category theory, the volume demonstrates its range of possible applications and should serve to stimulate our philosophical imaginations with respect to some of the open problems that it illuminates. Category theory is already being applied in physics, computer science and linguistics and it is easy to imagine that it will play a role in the work of more formally inclined philosophers whose interests touch on these areas. While it may be too early to tell whether category theory will have any lasting significance for philosophical inquiry, projects like Sica's encourages us to consider the possibilities.

Acknowledgments:

I am very grateful to Vladik Kreinovich for comments on an earlier version of this review.

References:

Awodey, S. (2006) *Category Theory.* Oxford: Oxford University Press

Awodey, S. and Reck, E. (2002) "Completeness and Categoricity I. Nineteen-Century Axiomatics to Twentieth-Century Metalogic", *History and Philosophy of Logic*, **23**, 1, 1–30.

Awodey, S. and Reck, E. (2002) "Completeness and Categoricity II. Twentieth-Century Metalogic to Twenty-first-Century Semantics", *History and Philosophy of Logic*, **23**, 2, 77–94.

Cole, J. C. (1973) Categories of Sets and Models of Set Theory, in *The Proceedings of the Bertrand Russell Memorial Logic Conference*, School of Mathematics, Leeds. 351-399

Feferman, S. (1998) *In the Light of Logic* (Oxford: Oxford University Press)

Goldblatt, R. (2006) *Topoi; The Categorial Analysis of Logic.* (Mineola, N.Y.: Dover Publications.)

Lawvere, F. W. and Stephen Hoel Schanuel (1997) *Conceptual Mathematics: A First Introduction to Categories.* Cambridge: Cambridge University Press

Mitchell, W. (1972) Boolean Topoi and the Theory of Sets. *Journal of Pure and Applied Algebra*, **9**, 143-164

Mac Lane, S. (1971) *Categories for the Working Mathematician*. New York: Springer

Marquis, J.-P. (2006) "Categories, Sets and the Nature of Mathematical Entities", in *The Age of Alternative Logics. Assessing philosophy of logic and mathematics today*, J. van Benthem, G. Heinzmann, Ph. Nabonnand, M. Rebuschi, H.Visser, eds., New York: Springer,181-192.

Peirce, B. (1991) *Basic Category Theory for Computer Scientists*, Cambridge: MIT Press.

Symons, J., J. Urenda and V. Kreinovich. (2007) Towards a General Description of Physical Invariance in Category Theory. *Journal of Uncertain Systems.* **1**. 3. 210-206